

# MAXIMALLY EMBEDDABLE COMPONENTS

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## Abstract

We investigate the partial orderings of the form  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ , where  $\mathbb{X} = \langle X, \rho \rangle$  is a countable binary relational structure and  $\mathbb{P}(\mathbb{X})$  the set of the domains of its isomorphic substructures and show that if the components of  $\mathbb{X}$  are maximally embeddable and satisfy an additional condition related to connectivity, then the poset  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is forcing equivalent to a finite power of  $(P(\omega)/\text{Fin})^+$ , or to  $(P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+$ , or to the direct product  $(P(\Delta)/\mathcal{ED}_{\text{fin}})^+ \times ((P(\omega)/\text{Fin})^+)^n$ , for some  $n \in \omega$ . In particular we obtain forcing equivalents of the posets of copies of countable equivalence relations, disconnected ultrahomogeneous graphs and some partial orderings.  
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## 1 Introduction

The posets of the form  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ , where  $\mathbb{X}$  is a relational structure and  $\mathbb{P}(\mathbb{X})$  the set of the domains of its isomorphic substructures, were investigated in [4]. In particular, a classification of countable binary structures related to the forcing-related properties of the posets of their copies is described in Diagram 1: for the structures from column *A* (resp. *B*; *D*) the corresponding posets are forcing equivalent to the trivial poset (resp. the Cohen forcing,  $\langle {}^{<\omega}2, \supset \rangle$ ; an  $\omega_1$ -closed atomless poset) and, for the structures from the class  $C_4$ , the posets of copies are forcing equivalent to the quotients of the form  $P(\omega)/\mathcal{I}$ , for some co-analytic tall ideal  $\mathcal{I}$ .

The aim of the paper is to investigate a subclass of column *D*, the class of structures  $\mathbb{X}$  for which the separative quotient  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is an  $\omega_1$ -closed and atomless poset (containing, for example, the class of all countable scattered linear orders [5]). Clearly, such a classification depends on the model of set theory in which we work. For example, under the CH all the structures from column *D* are in the same class (having the posets of copies forcing equivalent to the algebra  $P(\omega)/\text{Fin}$  without zero), but this is not true in, for example, the Mathias model.

Applying the main theorem of the paper, proved in Section 4, in Section 5 we obtain forcing equivalents of the posets of copies of countable equivalence relations, disconnected ultrahomogeneous graphs and some partial orderings.

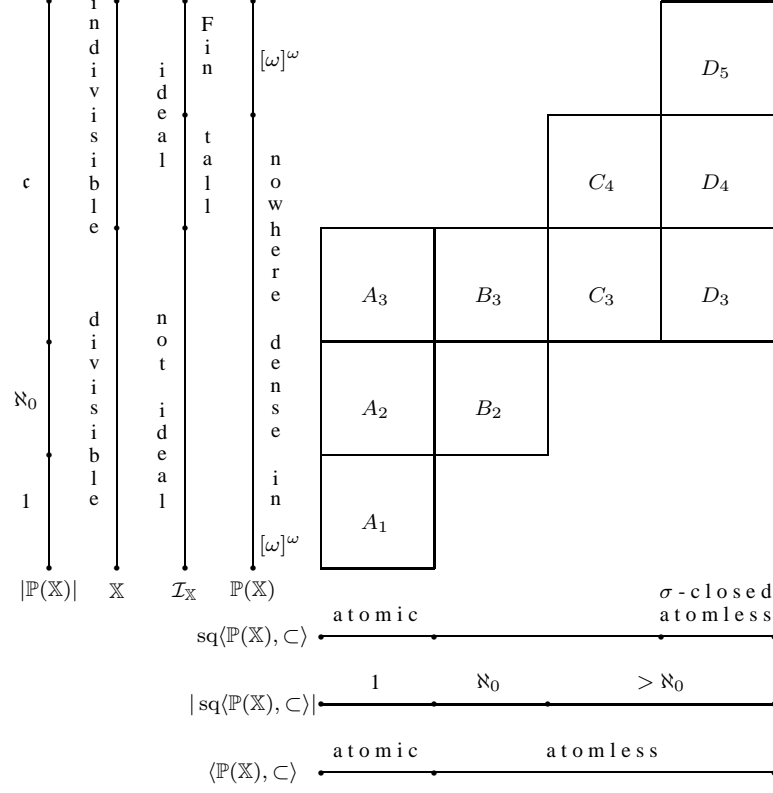


Diagram 1: Binary relations on countable sets

## 2 Preliminaries

Let  $\mathbb{P} = \langle P, \leq \rangle$  be a pre-order. Then  $p \in P$  is an *atom*, in notation  $p \in \text{At}(\mathbb{P})$ , iff each  $q, r \leq p$  are compatible (there is  $s \leq q, r$ ).  $\mathbb{P}$  is called *atomless* iff  $\text{At}(\mathbb{P}) = \emptyset$ ; *atomic* iff  $\text{At}(\mathbb{P})$  is dense in  $\mathbb{P}$ . If  $\kappa$  is a regular cardinal,  $\mathbb{P}$  is called  $\kappa$ -*closed* iff for each  $\gamma < \kappa$  each sequence  $\langle p_\alpha : \alpha < \gamma \rangle$  in  $P$ , such that  $\alpha < \beta \Rightarrow p_\beta \leq p_\alpha$ , has a lower bound in  $P$ .  $\omega_1$ -closed pre-orders are called  $\sigma$ -*closed*. Two pre-orders  $\mathbb{P}$  and  $\mathbb{Q}$  are called *forcing equivalent* iff they produce the same generic extensions.

A partial order  $\mathbb{P} = \langle P, \leq \rangle$  is called *separative* iff for each  $p, q \in P$  satisfying  $p \not\leq q$  there is  $r \leq p$  such that  $r \perp q$ . The *separative modification* of  $\mathbb{P}$  is the separative pre-order  $\text{sm}(\mathbb{P}) = \langle P, \leq^* \rangle$ , where  $p \leq^* q \Leftrightarrow \forall r \leq p \exists s \leq r \ s \leq q$ . The *separative quotient* of  $\mathbb{P}$  is the separative partial order  $\text{sq}(\mathbb{P}) = \langle P / \equiv^*, \sqsubseteq \rangle$ , where  $p \equiv^* q \Leftrightarrow p \leq^* q \wedge q \leq^* p$  and  $[p] \sqsubseteq [q] \Leftrightarrow p \leq^* q$ .

Let  $\text{Fin} = [\omega]^{<\omega}$  and  $\Delta = \{\langle m, n \rangle \in \mathbb{N} \times \mathbb{N} : n \leq m\}$ . Then the ideals  $\text{Fin} \times \text{Fin} \subset P(\omega \times \omega)$  and  $\mathcal{ED}_{\text{fin}} \subset P(\Delta)$  are defined by:

$$\begin{aligned} \text{Fin} \times \text{Fin} &= \{S \subset \omega \times \omega : \exists j \in \omega \forall i \geq j |S \cap (\{i\} \times \omega)| < \omega\} \text{ and} \\ \mathcal{ED}_{\text{fin}} &= \{S \subset \Delta : \exists r \in \mathbb{N} \forall m \in \mathbb{N} |S \cap (\{m\} \times \{1, 2, \dots, m\})| \leq r\}. \end{aligned}$$

By  $\mathfrak{h}(\mathbb{P})$  we denote the *distributivity number* of a poset  $\mathbb{P}$ . In particular, for  $n \in \mathbb{N}$ , let  $\mathfrak{h}_n = \mathfrak{h}((P(\omega)/\text{Fin})^+)^n$ ; thus  $\mathfrak{h} = \mathfrak{h}_1$ . The following statements will be used in the paper.

**Fact 2.1** (Folklore) If  $\mathbb{P}_i, i \in I$ , are  $\kappa$ -closed pre-orders, then  $\prod_{i \in I} \mathbb{P}_i$  is  $\kappa$ -closed.

**Fact 2.2** (Folklore) Let  $\mathbb{P}, \mathbb{Q}$  and  $\mathbb{P}_i, i \in I$ , be partial orderings. Then

- (a)  $\mathbb{P}$ ,  $\text{sm}(\mathbb{P})$  and  $\text{sq}(\mathbb{P})$  are forcing equivalent forcing notions;
- (b)  $\mathbb{P}$  is atomless iff  $\text{sm}(\mathbb{P})$  is atomless iff  $\text{sq}(\mathbb{P})$  is atomless;
- (c)  $\text{sm}(\mathbb{P})$  is  $\kappa$ -closed iff  $\text{sq}(\mathbb{P})$  is  $\kappa$ -closed;
- (d)  $\mathbb{P} \cong \mathbb{Q}$  implies that  $\text{sm } \mathbb{P} \cong \text{sm } \mathbb{Q}$  and  $\text{sq } \mathbb{P} \cong \text{sq } \mathbb{Q}$ ;
- (e)  $\text{sm}(\prod_{i \in I} \mathbb{P}_i) = \prod_{i \in I} \text{sm } \mathbb{P}_i$ ;
- (f)  $\text{sq}(\prod_{i \in I} \mathbb{P}_i) \cong \prod_{i \in I} \text{sq } \mathbb{P}_i$ .

**Fact 2.3** (Folklore) Let  $\mathbb{P}$  be an atomless separative pre-order. Then we have

- (a) If  $\omega_1 = \mathfrak{c}$  and  $\mathbb{P}$  is  $\omega_1$ -closed of size  $\mathfrak{c}$ , then  $\mathbb{P}$  is forcing equivalent to  $(\text{Coll}(\omega_1, \omega_1))^+$  or, equivalently, to  $(P(\omega)/\text{Fin})^+$ ;
- (b) If  $\mathfrak{t} = \mathfrak{c}$  and  $\mathbb{P}$  is  $\mathfrak{t}$ -closed of size  $\mathfrak{t}$ , then  $\mathbb{P}$  is forcing equivalent to  $(\text{Coll}(\mathfrak{t}, \mathfrak{t}))^+$  or, equivalently, to  $(P(\omega)/\text{Fin})^+$ .

**Fact 2.4** (a)  $\text{sm}(\langle [\omega]^\omega, \subset \rangle^n) = \langle [\omega]^\omega, \subset^* \rangle^n$  and  $\text{sq}(\langle [\omega]^\omega, \subset \rangle^n) = ((P(\omega)/\text{Fin})^+)^n$  are forcing equivalent,  $\mathfrak{t}$ -closed atomless pre-orders of size  $\mathfrak{c}$ .

- (b) (Shelah and Spinas [8])  $\text{Con}(\mathfrak{h}_{n+1} < \mathfrak{h}_n)$ , for each  $n \in \mathbb{N}$ .
- (c) (Szymański and Zhou [9])  $(P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+$  is an  $\omega_1$ -closed, but not  $\omega_2$ -closed atomless poset.
- (d) (Hernández-Hernández [3])  $\text{Con}(\mathfrak{h}((P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+) < \mathfrak{h})$ .
- (e) (Brendle [1])  $\text{Con}(\mathfrak{h}((P(\Delta)/\mathcal{ED}_{\text{fin}})^+) < \mathfrak{h})$ .

**Fact 2.5** If  $\langle P, \leq_P \rangle$  and  $\langle Q, \leq_Q \rangle$  are partial orderings and  $f : P \rightarrow Q$ , where

- (i)  $\forall p_1, p_2 \in P (p_1 \leq_P p_2 \Rightarrow f(p_1) \leq_Q f(p_2))$ ,
- (ii)  $\forall p_1, p_2 \in P (p_1 \perp_P p_2 \Rightarrow f(p_1) \perp_Q f(p_2))$ ,
- (iii)  $f[P] = Q$ ,

then  $\text{sq } \mathbb{P} \cong \text{sq } \mathbb{Q}$ .

**Proof.** We have  $\text{sm } \mathbb{P} = \langle P, \leq_P^* \rangle$ ,  $\text{sq } \mathbb{P} = \langle P/\equiv_P, \leq_P \rangle$ ,  $\text{sm } \mathbb{Q} = \langle Q, \leq_Q^* \rangle$  and  $\text{sq } \mathbb{Q} = \langle Q/\equiv_Q, \leq_Q \rangle$ , where for each  $p_1, p_2 \in P$  and each  $q_1, q_2 \in Q$

$$p_1 \leq_P^* p_2 \Leftrightarrow \forall p \leq_P p_1 \exists p' \leq_P p, p_2, \quad (1)$$

$$p_1 =_P p_2 \Leftrightarrow p_1 \leq_P^* p_2 \wedge p_2 \leq_P^* p_1 \quad \text{and} \quad [p_1] \sqsubseteq_P [p_2] \Leftrightarrow p_1 \leq_P^* p_2, \quad (2)$$

$$q_1 \leq_Q^* q_2 \Leftrightarrow \forall q \leq_Q q_1 \exists q' \leq_Q q, q_2, \quad (3)$$

$$q_1 =_Q q_2 \Leftrightarrow q_1 \leq_Q^* q_2 \wedge q_2 \leq_Q^* q_1 \quad \text{and} \quad [q_1] \sqsubseteq_Q [q_2] \Leftrightarrow q_1 \leq_Q^* q_2. \quad (4)$$

*Claim.*  $p_1 \leq_P^* p_2 \Leftrightarrow f(p_1) \leq_Q^* f(p_2)$ , for each  $p_1, p_2 \in P$ .

*Proof of Claim.* ( $\Rightarrow$ ) Let  $p_1 \leq_P^* p_2$ . According to (3) we prove

$$\forall q \leq_Q f(p_1) \exists q' \leq_Q q, f(p_2). \quad (5)$$

If  $q \leq_Q f(p_1)$  then, by (iii) there is  $p_3 \in P$  such that  $f(p_3) = q$ . By (ii) and since  $f(p_3) \leq_Q f(p_1)$ , there is  $p_4 \leq_P p_3, p_1$  and, by (1), there is  $p_5 \leq_P p_4, p_2$ , which, by (i), implies  $f(p_5) \leq_Q f(p_2)$ . Since  $p_5 \leq_P p_4 \leq_P p_3$  by (i) we have  $f(p_5) \leq_Q f(p_3) = q$  and  $q' = f(p_5)$  satisfies (5).

( $\Leftarrow$ ) Assuming (5) we prove that  $p_1 \leq_P^* p_2$ . If  $p \leq_P p_1$ , then, by (i),  $f(p) \leq_Q f(p_1)$  and, by (5), there is  $q' \leq_Q f(p), f(p_2)$  and, by (ii), there is  $p' \leq_P p, p_2$  and Claim is proved.

Now we show that  $\langle P/=P, \sqsubseteq_P \rangle \cong_F \langle Q/=Q, \sqsubseteq_Q \rangle$ , where  $F([p]) = [f(p)]$ .

By Claim, (2) and (4), for each  $p_1, p_2 \in P$  we have  $[p_1] = [p_2]$  iff  $p_1 =_P p_2$  iff  $p_1 \leq_P^* p_2 \wedge p_2 \leq_P^* p_1$  iff  $f(p_1) \leq_Q^* f(p_2) \wedge f(p_2) \leq_Q^* f(p_1)$  iff  $f(p_1) =_Q f(p_2)$  iff  $[f(p_1)] = [f(p_2)]$  iff  $F([p_1]) = F([p_2])$  and  $F$  is a well defined injection. By (iii), for  $[q] \in Q/=Q$  there is  $p \in P$  such that  $q = f(p)$ . Thus  $F([p]) = [f(p)] = [q]$  and  $F$  is a surjection.

By Claim, (2) and (4) again,  $[p_1] \sqsubseteq_P [p_2]$  iff  $p_1 \leq_P^* p_2$  iff  $f(p_1) \leq_Q^* f(p_2)$  iff  $[f(p_1)] \sqsubseteq_Q [f(p_2)]$  iff  $F([p_1]) \sqsubseteq_Q F([p_2])$ . Thus  $F$  is an isomorphism.  $\square$

### 3 Structures and posets of their copies

Let  $L = \{R\}$  be a relational language, where  $\text{ar}(R) = 2$ . An  $L$ -structure  $\mathbb{X} = \langle X, \rho \rangle$  is called a *countable structure* iff  $|X| = \omega$ . If  $A \subset X$ , then  $\langle A, \rho_A \rangle$  is a *substructure* of  $\mathbb{X}$ , where  $\rho_A = \rho \cap A^2$ . If  $\mathbb{Y} = \langle Y, \tau \rangle$  is an  $L$ -structure too, a map  $f : X \rightarrow Y$  is called an *embedding* (we write  $\mathbb{X} \hookrightarrow_f \mathbb{Y}$ ) iff it is an injection and  $\langle x_1, x_2 \rangle \in \rho \Leftrightarrow \langle f(x_1), f(x_2) \rangle \in \tau$ , for each  $\langle x_1, x_2 \rangle \in X^2$ . If  $\mathbb{X}$  embeds in  $\mathbb{Y}$  we write  $\mathbb{X} \hookrightarrow \mathbb{Y}$ . Let  $\text{Emb}(\mathbb{X}, \mathbb{Y}) = \{f : \mathbb{X} \hookrightarrow_f \mathbb{Y}\}$  and, in particular,  $\text{Emb}(\mathbb{X}) = \{f : \mathbb{X} \hookrightarrow_f \mathbb{X}\}$ . If, in addition,  $f$  is a surjection, it is an *isomorphism* (we write  $\mathbb{X} \cong_f \mathbb{Y}$ ) and the structures  $\mathbb{X}$  and  $\mathbb{Y}$  are *isomorphic*, in notation  $\mathbb{X} \cong \mathbb{Y}$ .  $\mathbb{X}$  and  $\mathbb{Y}$  are *equimorphic* iff  $\mathbb{X} \hookrightarrow \mathbb{Y}$  and  $\mathbb{Y} \hookrightarrow \mathbb{X}$ . According to [2] a relational structure  $\mathbb{X}$  is: *p-monomorphic* iff all its substructures of size  $p$  are isomorphic; *indivisible* iff for each partition  $X = A \cup B$  we have  $\mathbb{X} \hookrightarrow A$  or  $\mathbb{X} \hookrightarrow B$ .

If  $\mathbb{X}_i = \langle X_i, \rho_i \rangle$ ,  $i \in I$ , are  $L$ -structures and  $X_i \cap X_j = \emptyset$ , for  $i \neq j$ , then the structure  $\bigcup_{i \in I} \mathbb{X}_i = \langle \bigcup_{i \in I} X_i, \bigcup_{i \in I} \rho_i \rangle$  is the *union* of the structures  $\mathbb{X}_i$ ,  $i \in I$ .

Let  $\langle X, \rho \rangle$  be an  $L$ -structure and  $\rho_{rst}$  the minimal equivalence relation on  $X$  containing  $\rho$  (the transitive closure of the relation  $\rho_{rs} = \Delta_X \cup \rho \cup \rho^{-1}$  given by  $x \rho_{rst} y$  iff there are  $n \in \mathbb{N}$  and  $z_0 = x, z_1, \dots, z_n = y$  such that  $z_i \rho_{rs} z_{i+1}$ , for each  $i < n$ ). For  $x \in X$  the corresponding equivalence class will be denoted by  $[x]$  and called the *component* of  $\langle X, \rho \rangle$  containing  $x$ . The structure  $\langle X, \rho \rangle$  will be called *connected* iff it has only one component. It is easy to prove (see [4]) that  $\langle X, \rho \rangle = \langle \bigcup_{x \in X} [x], \bigcup_{x \in X} \rho_{[x]} \rangle$  is the unique representation of  $\langle X, \rho \rangle$  as a disjoint union of connected relations.

Here we investigate the partial orders of the form  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ , where  $\mathbb{X} = \langle X, \rho \rangle$  is an  $L$ -structure and  $\mathbb{P}(\mathbb{X})$  the set of its isomorphic substructures, that is

$$\mathbb{P}(\mathbb{X}) = \{A \subset X : \langle A, \rho_A \rangle \cong \mathbb{X}\} = \{f[X] : f \in \text{Emb}(\mathbb{X})\}.$$

More generally, if  $\mathbb{X} = \langle X, \rho \rangle$  and  $\mathbb{Y} = \langle Y, \tau \rangle$  are two  $L$ -structures we define  $\mathbb{P}(\mathbb{X}, \mathbb{Y}) = \{B \subset Y : \langle B, \tau_B \rangle \cong \mathbb{X}\} = \{f[X] : f \in \text{Emb}(\mathbb{X}, \mathbb{Y})\}$ . Also let  $\mathcal{I}_{\mathbb{X}} = \{S \subset X : \neg \exists A \in \mathbb{P}(\mathbb{X}) A \subset S\}$ . We will use the following statements.

**Fact 3.1 ([4])** For each relational structure  $\mathbb{X}$  we have:  $|\text{sq}(\mathbb{P}(\mathbb{X}), \subset)| \geq \aleph_0$  iff the poset  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is atomless iff  $\mathbb{P}(\mathbb{X})$  contains two incompatible elements.

**Fact 3.2 ([4])** A structure  $\mathbb{X}$  is indivisible iff  $\mathcal{I}_{\mathbb{X}}$  is an ideal in  $P(X)$ . Then

- (a)  $\text{sm}(\mathbb{P}(\mathbb{X}), \subset) = \langle \mathbb{P}(\mathbb{X}), \subset_{\mathcal{I}_{\mathbb{X}}} \rangle$ , where  $A \subset_{\mathcal{I}_{\mathbb{X}}} B \Leftrightarrow A \setminus B \in \mathcal{I}_{\mathbb{X}}$ ;
- (b)  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset)$  is isomorphic to a dense subset of  $\langle (P(X)/\equiv_{\mathcal{I}_{\mathbb{X}}})^+, \leq_{\mathcal{I}_{\mathbb{X}}} \rangle$ . Hence the poset  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is forcing equivalent to  $(P(X)/\mathcal{I}_{\mathbb{X}})^+$ .
- (c) If  $\mathbb{X}$  is countable, then  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is an atomless partial order of size  $\mathfrak{c}$ .

**Fact 3.3 ([4])** Let  $\mathbb{X}_i = \langle X_i, \rho_i \rangle$ ,  $i \in I$ , and  $\mathbb{Y}_j = \langle Y_j, \sigma_j \rangle$ ,  $j \in J$ , be two families of disjoint connected  $L$ -structures and  $\mathbb{X}$  and  $\mathbb{Y}$  their unions. Then

- (a)  $F : \mathbb{X} \hookrightarrow \mathbb{Y}$  iff  $F = \bigcup_{i \in I} g_i$ , where  $f : I \rightarrow J$ ,  $g_i : \mathbb{X}_i \hookrightarrow \mathbb{Y}_{f(i)}$ ,  $i \in I$ , and

$$\forall \{i_1, i_2\} \in [I]^2 \quad \forall x_{i_1} \in X_{i_1} \quad \forall x_{i_2} \in X_{i_2} \quad \neg g_{i_1}(x_{i_1}) \sigma_{rs} g_{i_2}(x_{i_2}); \quad (6)$$

- (b)  $C \in \mathbb{P}(\mathbb{X})$  iff  $C = \bigcup_{i \in I} g_i[X_i]$ , where  $f : I \rightarrow I$ ,  $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_{f(i)}$ ,  $i \in I$ , and

$$\forall \{i, j\} \in [I]^2 \quad \forall x \in X_i \quad \forall y \in X_j \quad \neg g_i(x) \rho_{rs} g_j(y). \quad (7)$$

**Fact 3.4 ([4])** If  $\mathbb{X}$  and  $\mathbb{Y}$  are equimorphic structures, then the posets  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  and  $\langle \mathbb{P}(\mathbb{Y}), \subset \rangle$  are forcing equivalent.

**Fact 3.5 (Pouzet [7])** If  $p \leq |X|$  and  $\mathbb{X}$  is  $p$ -monomorphic, then  $\mathbb{X}$  is  $r$ -monomorphic for each  $r \leq \min\{p, |X| - p\}$ . (See also [2], p. 259.)

## 4 Structures with maximally embeddable components

**Theorem 4.1** Let  $\mathbb{X}_i = \langle X_i, \rho_{X_i} \rangle$ ,  $i \in I$ , be the components of a countable  $L$ -structure  $\mathbb{X} = \langle X, \rho \rangle$  and, for all  $i, j \in I$ , let

- (i)  $\mathbb{P}(\mathbb{X}_i, \mathbb{X}_j) = [\mathbb{X}_j]^{|\mathbb{X}_i|}$  (the components of  $\mathbb{X}$  are maximally embeddable),
- (ii)  $\forall A, B \in [\mathbb{X}_j]^{|\mathbb{X}_i|} \exists a \in A \exists b \in B \ a \rho_{rs} b$ .

If  $N = \{|X_i| : i \in I\}$ ,  $N_{\text{fin}} = N \setminus \{\omega\}$ ,  $I_\kappa = \{i \in I : |X_i| = \kappa\}$ , for  $\kappa \in N$ ,  $|I_\omega| = \mu$  and  $Y = \bigcup_{i \in I \setminus I_\omega} X_i$ , then we have

(a)  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset)$  is an  $\omega_1$ -closed atomless poset of size  $\mathfrak{c}$ . In addition, it is isomorphic (and, hence, the poset  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is forcing equivalent) to the poset

$$(P(\omega)/\text{Fin})^+{}^\mu \quad \text{if } 1 \leq \mu < \omega, \ |N_{\text{fin}}| < \omega \text{ and } |Y| < \omega, \quad (\text{a1})$$

$$((P(\omega)/\text{Fin})^+{}^\mu)^{\mu+1} \quad \text{if } 0 \leq \mu < \omega, \ |N_{\text{fin}}| < \omega \text{ and } |Y| = \omega, \quad (\text{a2})$$

$$\mathbb{P} \times ((P(\omega)/\text{Fin})^+{}^\mu) \quad \text{if } 0 \leq \mu < \omega, \ |N_{\text{fin}}| = \omega, \quad (\text{a3})$$

$$(P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+ \quad \text{if } \mu = \omega, \quad (\text{a4})$$

where  $\mathbb{P}$  is an  $\omega_1$ -closed atomless poset, forcing equivalent to  $(P(\Delta)/\mathcal{ED}_{\text{fin}})^+$ .

(b) For some forcing related cardinal invariants of the poset  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  we have

If $\mathbb{X}$ satisfies	$\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is forcing equivalent to	$\text{sq}(\mathbb{P}(\mathbb{X}), \subset)$ is	$\text{ZFC} \vdash \text{sq}(\mathbb{P}(\mathbb{X}), \subset)$ is $\mathfrak{h}$ -distributive
$\mu < \omega \wedge  N_{\text{fin}}  < \omega$	$((P(\omega)/\text{Fin})^+)^n$ , for some $n \in \mathbb{N}$	$\mathfrak{t}$ -closed	yes iff $n = 1$
$\mu < \omega \wedge  N_{\text{fin}}  = \omega$	$(P(\Delta)/\mathcal{ED}_{\text{fin}})^+ \times ((P(\omega)/\text{Fin})^+{}^\mu)$	$\omega_1$ -closed	no
$\mu = \omega$	$(P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+$	$\omega_1$ but not $\omega_2$ -closed	no

where  $n = 1$  iff  $N \in [\mathbb{N}]^{<\omega} \vee (|Y| < \omega \wedge \mu = 1)$ .

(c)  $\mathbb{X}$  is indivisible iff  $N \in [\mathbb{N}]^\omega$  or  $N = \{1\}$  or  $|I| = 1$  or  $|I_\omega| = \omega$ .

A proof of the theorem, given at the end of this section, is based on the following five claims.

**Claim 4.2**  $C \in \mathbb{P}(\mathbb{X})$  iff there is an injection  $f : I \rightarrow I$  and there are  $C_i \in [X_{f(i)}]^{|\mathbb{X}_i|}$ ,  $i \in I$ , such that  $C = \bigcup_{i \in I} C_i$ .

**Proof.**  $(\Rightarrow)$  Let  $C \in \mathbb{P}(\mathbb{X})$ . By Fact 3.3(b) there are functions  $f : I \rightarrow I$  and  $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_{f(i)}$ ,  $i \in I$ , satisfying (7) and such that  $C = \bigcup_{i \in I} g_i[X_i]$ . By (7) and (ii),  $f$  is an injection. Since  $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_{f(i)}$  we have  $C_i = g_i[X_i] \in \mathbb{P}(\mathbb{X}_i, \mathbb{X}_{f(i)}) = [\mathbb{X}_{f(i)}]^{|\mathbb{X}_i|}$ .

$(\Leftarrow)$  Suppose that  $f$  and  $C_i$ ,  $i \in I$ , satisfy the assumptions. Since  $[\mathbb{X}_{f(i)}]^{|\mathbb{X}_i|} = \mathbb{P}(\mathbb{X}_i, \mathbb{X}_{f(i)})$  there are  $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_{f(i)}$ ,  $i \in I$ , such that  $C_i = g_i[X_i]$ . Since  $f$

is an injection, for different  $i, j \in I$  the sets  $g_i[X_i]$  and  $g_j[X_j]$  are in different components of  $\mathbb{X}$  and, hence, we have (7). By Fact 3.3(b),  $C \in \mathbb{P}(\mathbb{X})$ .  $\square$

We continue the proof considering the following cases and subcases.

1.  $N \subset \mathbb{N}$ , with subcases  $N \in [\mathbb{N}]^\omega$  (Claim 4.3) and  $N \in [\mathbb{N}]^{<\omega}$  (Claim 4.4);
2.  $N \not\subset \mathbb{N}$ , with subcases  $|I_\omega| < \omega$  (Claim 4.5) and  $|I_\omega| = \omega$  (Claim 4.6).

**Case 1:**  $N \subset \mathbb{N}$ .

**Claim 4.3 (Case 1.1)** If  $N \in [\mathbb{N}]^\omega$ , then

- (a)  $\mathbb{X}$  is an indivisible structure;
- (b)  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset)$  is an  $\omega_1$ -closed atomless poset;
- (c) The structures  $\mathbb{X}_i$ ,  $i \in I$ , are either full relations or complete graphs or reflexive or irreflexive linear orderings;
- (d) There are structures  $\mathbb{X}_n$ ,  $n \in \mathbb{N} \setminus N$ , such that  $|X_n| = n$  and that the extended family  $\{\mathbb{X}_i : i \in I\} \cup \{\mathbb{X}_n : n \in \mathbb{N} \setminus N\}$  satisfies (i) and (ii);
- (e) The poset  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is forcing equivalent to  $(P(\Delta)/\mathcal{ED}_{\text{fin}})^+$ .

**Proof.** Clearly,  $N \in [\mathbb{N}]^\omega$  implies that  $|I| = \omega$ . First we prove

$$S \in \mathcal{I}_{\mathbb{X}} \Leftrightarrow \exists n \in \omega \forall i \in I |S \cap X_i| \leq n. \quad (8)$$

( $\Rightarrow$ ) Here, for convenience, we assume that  $I = \omega$ . Suppose that for each  $n \in \omega$  there is  $i \in I$  such that  $|S \cap X_i| > n$ . Then  $I_{>n}^S = \{i \in \omega : |S \cap X_i| > n\}$ ,  $n \in \omega$ , are infinite sets. By recursion we define sequences  $\langle i_k : k \in \omega \rangle$  in  $\omega$  and  $\langle C_k : k \in \omega \rangle$  in  $P(X)$  such that for each  $k, l \in \omega$

- (i)  $k < l \Rightarrow i_k < i_l$ ,
- (ii)  $C_k \in [S \cap X_{i_k}]^{|X_k|}$ .

Suppose that the sequences  $i_0, \dots, i_k$  and  $C_0, \dots, C_k$  satisfy (i) and (ii). Since  $|I_{>|X_{k+1}|}^S| = \omega$  there is  $i_{k+1} = \min\{i > i_k : |S \cap X_i| > |X_{k+1}|\}$  so  $|S \cap X_{i_{k+1}}| > |X_{k+1}|$ , we choose  $C_{k+1} \in [S \cap X_{i_{k+1}}]^{|X_{k+1}|}$  and the recursion works.

By (i) the function  $f : I \rightarrow I$  defined by  $f(k) = i_k$  is an injection. By (ii) we have  $C_k \in [X_{f(k)}]^{|X_k|}$  and, by Claim 4.2  $C = \bigcup_{k \in \omega} C_k \in \mathbb{P}(\mathbb{X})$ . Since  $C \subset S$  we have  $S \notin \mathcal{I}_{\mathbb{X}}$ .

( $\Leftarrow$ ) Suppose that  $C \in \mathbb{P}(\mathbb{X})$ , where  $C \subset S$ . By Claim 4.2 there are an injection  $f : I \rightarrow I$  and  $C_i \in [X_{f(i)}]^{|X_i|}$ ,  $i \in I$ , such that  $C = \bigcup_{i \in I} C_i$ . For  $n \in \omega$  there is  $i_0 \in I$  such that  $|X_{i_0}| > n$  and, hence,  $C_{i_0} \in [X_{f(i_0)}]^{|X_{i_0}|}$ , which implies  $|X_{f(i_0)} \cap S| \geq |C_{i_0}| > n$ . (8) is proved.

(a) Suppose that  $X = C \cup D$  is a partition, where  $C, D \in \mathcal{I}_{\mathbb{X}}$ . Then, by (8), there are  $m, n \in \omega$  such that  $|C \cap X_i| \leq m$  and  $|D \cap X_i| \leq n$ , for each  $i \in I$ . Hence for each  $i \in I$  we have  $|X_i| = |(X_i \cap C) \cup (X_i \cap D)| \leq m + n$ , which is impossible since, by the assumption,  $N \in [\mathbb{N}]^\omega$ .

(b) By Facts 2.2(b) and (c) it is sufficient to show that  $\text{sm}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is an  $\omega_1$ -closed and atomless pre-order. Let  $\text{sm}\langle \mathbb{P}(\mathbb{X}), \subset \rangle = \langle \mathbb{P}(\mathbb{X}), \leq \rangle$ . By Fact 3.2 and (a) for each  $A, B \in \mathbb{P}(\mathbb{X})$  we have  $A \leq B$  iff  $A \setminus B \in \mathcal{I}_{\mathbb{X}}$  and, by (8),

$$A \leq B \Leftrightarrow \exists n \in \mathbb{N} \ \forall i \in I \ |A \setminus B \cap X_i| \leq n. \quad (9)$$

Let  $A_n \in \mathbb{P}(\mathbb{X})$ ,  $n \in \omega$ , and  $A_{n+1} \leq A_n$ , for all  $n \in \omega$ . We will find  $A \in \mathbb{P}(\mathbb{X})$  such that  $A \leq A_n$ , for all  $n \in \omega$ , that is, by (9),

$$\forall n \in \omega \ \exists m \in \mathbb{N} \ \forall i \in I \ |A \setminus A_n \cap X_i| \leq m. \quad (10)$$

By recursion we define a sequence  $\langle i_r : r \in \omega \rangle$  in  $I$  such that for each  $r, s \in \omega$

- (i)  $r \neq s \Rightarrow i_r \neq i_s$ ,
- (ii)  $|A_0 \cap A_1 \cap \dots \cap A_r \cap X_{i_r}| > r$ .

First we choose  $i_0$  such that  $|A_0 \cap X_{i_0}| > 0$ . Let the sequence  $i_0, \dots, i_r$  satisfy (i) and (ii). For each  $k \leq r$  we have  $A_{k+1} \leq A_k$  and, by (9), there is  $m_k \in \omega$  such that  $\forall i \in I \ |A_{k+1} \setminus A_k \cap X_i| \leq m_k$ . Thus

$$\forall i \in I \ \forall k \leq r \ |A_{k+1} \setminus A_k \cap X_i| \leq m_k. \quad (11)$$

Since  $A_{r+1} \in \mathbb{P}(\mathbb{X})$  and  $N \in [\mathbb{N}]^\omega$ , by Claim 4.2 the set

$$J_{r+1} = \{i \in I : |A_{r+1} \cap X_i| > (\sum_{k \leq r} m_k) + r + 1\} \quad (12)$$

is infinite and we choose

$$i_{r+1} \in J_{r+1} \setminus \{i_0, \dots, i_r\}. \quad (13)$$

Then (i) is true. Clearly,  $A_{r+1} \subset (\bigcap_{k=0}^{r+1} A_k) \cup \bigcup_{k=0}^r (A_{k+1} \setminus A_k)$  and, hence,  $A_{r+1} \cap X_{i_{r+1}} \subset (\bigcap_{k=0}^{r+1} A_k \cap X_{i_{r+1}}) \cup \bigcup_{k=0}^r (A_{k+1} \setminus A_k \cap X_{i_{r+1}})$ . So, by (11)-(13)  $(\sum_{k \leq r} m_k) + r + 1 < |A_{r+1} \cap X_{i_{r+1}}| \leq |\bigcap_{k=0}^{r+1} A_k \cap X_{i_{r+1}}| + \sum_{k \leq r} m_k$ , which implies  $|A_0 \cap \dots \cap A_r \cap A_{r+1} \cap X_{i_{r+1}}| > r + 1$  and (ii) is true. The recursion works.

Let  $S = \bigcup_{r \in \omega} (A_0 \cap A_1 \cap \dots \cap A_r \cap X_{i_r})$ . By (i), (ii) and (8) we have  $S \notin \mathcal{I}_{\mathbb{X}}$  and, hence, there is  $A \in \mathbb{P}(\mathbb{X})$  such that  $A \subset S$ . We prove (10). For  $n \in \omega$  we have  $A \setminus A_n \subset S \setminus A_n \subset \bigcup_{r < n} (A_0 \cap A_1 \cap \dots \cap A_r \cap X_{i_r}) \subset \bigcup_{r < n} X_{i_r}$ , thus  $|A \setminus A_n| = m$ , for some  $m \in \omega$  and, hence,  $|A \setminus A_n \cap X_i| \leq m$ , for each  $i \in I$ .

So  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is  $\omega_1$ -closed. By (a) and Facts 3.2(c) and 2.2(b) it is atomless.

(c) Since  $N \in [\mathbb{N}]^\omega$ , there are  $i_0, i_1 \in I$  such that  $|X_{i_0}| \geq 3$  and  $|X_{i_1}| \geq |X_{i_0}| + 3$ . By (i) we have  $\mathbb{P}(\mathbb{X}_{i_0}, \mathbb{X}_{i_1}) = [\mathbb{X}_{i_1}]^{|\mathbb{X}_{i_0}|}$  and, hence, the structure  $\mathbb{X}_{i_1}$  is  $|X_{i_0}|$ -monomorphic. Since  $|X_{i_1}| - |X_{i_0}| \geq 3$  we have  $\min\{|X_{i_0}|, |X_{i_1}| - |X_{i_0}|\} \geq 3$  and, by Fact 3.5,

$$\forall r \leq 3 \ (\mathbb{X}_{i_1} \text{ is } r\text{-monomorphic}). \quad (14)$$



Let  $\{y_1, y_2, y_3\} \in [X_{i_1}]^3$  and, for  $r \in \{1, 2, 3\}$ , let  $\mathbb{Y}_r = \langle Y_r, \tau_r \rangle$ , where  $Y_r = \{y_k : k \leq r\}$  and  $\tau_r = (\rho_{i_1})_{Y_r}$ . We prove

$$\forall i \in I \quad \forall r \leq \min\{3, |X_i|\} \quad \forall A \in [X_i]^r \quad \langle A, (\rho_i)_A \rangle \cong \mathbb{Y}_r. \quad (15)$$

If  $|X_i| \geq |X_{i_1}|$ , let  $A \subset B \in [X_i]^{|X_{i_1}|}$ . By (i) there exists an isomorphism  $f : \langle B, (\rho_i)_B \rangle \rightarrow \mathbb{X}_{i_1}$  and, by (14) we have  $\langle A, (\rho_i)_A \rangle \cong \langle f[A], (\rho_{i_1})_{f[A]} \rangle \cong \mathbb{Y}_r$ .

If  $|X_i| < |X_{i_1}|$  then, by (i), there exists an isomorphism  $f : \mathbb{X}_i \rightarrow \mathbb{X}_{i_1}$  and by (14) we have  $\langle A, (\rho_i)_A \rangle \cong \langle f[A], (\rho_{i_1})_{f[A]} \rangle \cong \mathbb{Y}_r$ . Thus (15) is true.

Clearly we have  $\tau_1 = \emptyset$  or  $\tau_1 = \{\langle y_1, y_1 \rangle\}$ .

First, suppose that  $\tau_1 = \emptyset$ . Then by (15), for each  $i \in I$  we have

$$\forall x \in X_i \quad \neg x \rho_i x, \quad (16)$$

that is, all relations  $\rho_i, i \in I$ , are irreflexive. Suppose that  $\tau_2 \cap \{\langle y_1, y_2 \rangle, \langle y_2, y_1 \rangle\} = \emptyset$ . Then by (15) we would have  $\rho_{i_1} = \emptyset$  and  $\mathbb{X}_{i_1}$  would be a disconnected structure, which is not true. Thus  $\tau_2 \cap \{\langle y_1, y_2 \rangle, \langle y_2, y_1 \rangle\} \neq \emptyset$ .

Thus, if  $\langle y_1, y_2 \rangle, \langle y_2, y_1 \rangle \in \tau_2$ , then by (15), for each  $i \in I$  we have

$$\forall \{x, y\} \in [X_i]^2 \quad (x \rho_i y \wedge y \rho_i x) \quad (17)$$

and, hence,  $\mathbb{X}_i$  is a complete graph.

Otherwise, if  $|\tau_2 \cap \{\langle y_1, y_2 \rangle, \langle y_2, y_1 \rangle\}| = 1$  then, by (15), for each  $i \in I$  we have

$$\forall \{x, y\} \in [X_i]^2 \quad (x \rho_i y \vee y \rho_i x) \quad (18)$$

and, hence,  $\mathbb{X}_i$  is a tournament. Thus  $\mathbb{Y}_3$  is a tournament with three nodes and, hence,  $\mathbb{Y}_3 \cong C_3 = \langle \{1, 2, 3\}, \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle\} \rangle$  (the oriented circle graph) or  $\mathbb{Y}_3 \cong L_3 = \langle \{1, 2, 3\}, \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle\} \rangle$  (the transitive triple, the strict linear order of size 3). But  $\mathbb{Y}_3 \cong C_3$  would imply that  $\mathbb{X}_{i_1}$  contains a four element tournament having all substructures of size 3 isomorphic to  $C_3$ , which is impossible. Thus  $\mathbb{Y}_3 \cong L_3$  which, together with (15), (16) and (18) implies that all relations  $\rho_i, i \in I$  are transitive, so  $\mathbb{X}_i, i \in I$ , are strict linear orders.

If  $\tau_1 = \{\langle y_1, y_1 \rangle\}$  then using the same arguments we show that the structures  $\mathbb{X}_i, i \in I$ , are either full relations or reflexive linear orders.

(d) follows from (c). Namely, if, for example,  $\mathbb{X}_i$  are complete graphs, then  $\mathbb{X}_n$  are complete graphs of size  $n$ .

(e) Let  $N = \{n_k : k \in \mathbb{N}\}$ , where  $n_1 < n_2 < \dots$  and let  $\mathbb{X}_n, n \in \mathbb{N} \setminus N$ , be the structures from (d). W.l.o.g. suppose that  $I_{n_k} = \{n_k\} \times \{1, 2, \dots, |I_{n_k}|\}$ , if  $|I_{n_k}| \in \mathbb{N}$ , and  $I_{n_k} = \{n_k\} \times \mathbb{N}$ , if  $|I_{n_k}| = \omega$ . Then  $I \subset \mathbb{N} \times \mathbb{N}$  and  $X = \bigcup_{k \in \mathbb{N}} \bigcup_{\langle n_k, r \rangle \in I_{n_k}} X_{\langle n_k, r \rangle}$ . For  $l \in \mathbb{N}$ , let  $\mathbb{Y}_l = \langle Y_l, \rho_l \rangle$  be defined by

$$\mathbb{Y}_l = \begin{cases} \mathbb{X}_l & \text{if } l \in \mathbb{N} \setminus N, \\ \mathbb{X}_{\langle n_k, 1 \rangle} & \text{if } l = n_k, \text{ for a } k \in \mathbb{N}. \end{cases}$$

and let  $\mathbb{Y} = \langle \bigcup_{l \in \mathbb{N}} Y_l, \bigcup_{l \in \mathbb{N}} \rho_l \rangle$ . We prove that  $\mathbb{X} \hookrightarrow \mathbb{Y}$  and  $\mathbb{Y} \hookrightarrow \mathbb{X}$ .

$\mathbb{Y} \hookrightarrow \mathbb{X}$ . Let  $f : \mathbb{N} \rightarrow I$ , where  $f(l) = \langle n_l, 1 \rangle$ . Since  $n_1 < n_2 < \dots$  we have  $|Y_l| = l \leq n_l = |X_{\langle n_l, 1 \rangle}| = |X_{f(l)}|$  and, since the extended family of structures satisfies (i), there is  $g_l : \mathbb{Y}_l \hookrightarrow \mathbb{X}_{f(l)}$ . Since  $f$  is an injection, the sets  $g_l[Y_l]$ ,  $l \in \mathbb{N}$ , are in different components of  $\mathbb{X}$  and, hence, condition (6) is satisfied. Thus, by Fact 3.3(a),  $F = \bigcup_{l \in \mathbb{N}} g_l : \mathbb{Y} \hookrightarrow \mathbb{X}$ .

$\mathbb{X} \hookrightarrow \mathbb{Y}$ . Let  $\mathbb{N} = \bigcup_{k \in \mathbb{N}} J_k$  be a partition, where  $|J_k| = \omega$ , for each  $k \in \mathbb{N}$ , and let  $Z_k = \bigcup_{\langle n_k, r \rangle \in I_{n_k}} X_{\langle n_k, r \rangle}$  and  $T_k = \bigcup_{l \in J_k} Y_l$ , for  $k \in \mathbb{N}$ . Now  $|I_{n_k}| \leq \omega = |J_k|$  and for  $l \geq n_k$  we have  $|X_{\langle n_k, r \rangle}| = n_k \leq l = |Y_l|$ . Hence there is an injection  $f_k : I_{n_k} \rightarrow J_k \setminus n_k$  and, since the extended family satisfies (i), there are embeddings  $g_{\langle n_k, r \rangle} : \mathbb{X}_{\langle n_k, r \rangle} \hookrightarrow \mathbb{Y}_{f_k(\langle n_k, r \rangle)}$ , for  $\langle n_k, r \rangle \in I_{n_k}$ . Thus,  $f = \bigcup_{k \in \mathbb{N}} f_k : I \rightarrow \mathbb{N}$  and condition (6) is satisfied so, by Fact 3.3,  $F = \bigcup_{k \in \mathbb{N}} \bigcup_{\langle n_k, r \rangle \in I_{n_k}} g_{\langle n_k, r \rangle}$  embeds  $\mathbb{X} = \bigcup_{k \in \mathbb{N}} \bigcup_{\langle n_k, r \rangle \in I_{n_k}} \mathbb{X}_{\langle n_k, r \rangle}$  into  $\mathbb{Y} = \bigcup_{k \in \mathbb{N}} \bigcup_{l \in J_k} \mathbb{Y}_l$ .

Now, by Fact 3.4, the posets  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  and  $\langle \mathbb{P}(\mathbb{Y}), \subset \rangle$  are forcing equivalent. W.l.o.g. suppose that  $Y_l = \{l\} \times \{1, 2, \dots, l\} \subset \mathbb{N} \times \mathbb{N}$ . Then  $Y = \Delta = \{\langle l, m \rangle \in \mathbb{N} \times \mathbb{N} : m \leq l\}$  and, by (8),  $S \in \mathcal{I}_{\mathbb{Y}}$  iff  $\exists n \in \mathbb{N} \forall l \in \mathbb{N} |S \cap Y_l| \leq n$  iff  $S \in \mathcal{ED}_{\text{fin}}$ . Thus  $\mathcal{I}_{\mathbb{Y}} = \mathcal{ED}_{\text{fin}}$  and, by Claim 4.3(a) and Fact 3.2(b),  $\langle \mathbb{P}(\mathbb{Y}), \subset \rangle$  is forcing equivalent to  $(P(Y)/\mathcal{I}_{\mathbb{Y}})^+$ , that is to  $(P(\Delta)/\mathcal{ED}_{\text{fin}})^+$ .  $\square$

**Claim 4.4 (Case 1.2)** If  $N \in [\mathbb{N}]^{<\omega}$ , then we have

- (a)  $\text{sq} \langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong (P(\omega)/\text{Fin})^+$ ;
- (b)  $\mathbb{X}$  is an indivisible structure iff  $m = 1$ , where  $m = \max N$ .

**Proof.** (a) Case A:  $|I_m| = \omega$ . For  $S \subset X$  let  $I_m^S = \{i \in I_m : X_i \subset S\}$ . First we prove

$$S \in \mathcal{I}_{\mathbb{X}} \Leftrightarrow |I_m^S| < \omega. \quad (19)$$

Let  $S \notin \mathcal{I}_{\mathbb{X}}$  and  $C \subset S$ , where  $C \in \mathbb{P}(\mathbb{X})$ . By Claim 4.2 there are an injection  $f : I \rightarrow I$  and  $C_i \in [X_{f(i)}]^{X_i}$ ,  $i \in I$ , such that  $C = \bigcup_{i \in I} C_i$ . For  $i \in I_m$  we have  $|X_i| = m$  and, since  $C_i \in [X_{f(i)}]^m$ , we have  $|X_{f(i)}| = m$  and  $C_i = X_{f(i)} \subset S$ . Thus  $f(i) \in I_m^S$ , for each  $i \in I_m$  which, since  $f$  is one-to-one, implies  $|I_m^S| = \omega$ .

Suppose that  $|I_m^S| = \omega$  and let  $f : I \rightarrow I_m^S$  be a bijection. For  $i \in I$  we have  $X_{f(i)} \subset S$  and  $|X_i| \leq m = |X_{f(i)}|$  and we choose  $C_i \in [X_{f(i)}]^{X_i}$ . Now  $C = \bigcup_{i \in I} C_i \subset S$  and, by Claim 4.2,  $C \in \mathbb{P}(\mathbb{X})$ . Thus  $S \notin \mathcal{I}_{\mathbb{X}}$  and (19) is proved.

W.l.o.g. we assume that  $I_m = \omega$ . By (19), for  $A \in \mathbb{P}(\mathbb{X})$  we have  $I_m^A \in [\omega]^\omega$  and we show that the posets  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  and  $\langle [\omega]^\omega, \subset \rangle$  and the mapping  $f : \mathbb{P}(\mathbb{X}) \rightarrow [\omega]^\omega$  defined by  $f(A) = I_m^A$  satisfy the assumptions of Fact 2.5. Clearly,  $A \subset B$  implies  $I_m^A \subset I_m^B$  and (i) is true. If  $A$  and  $B$  are incompatible elements of  $\mathbb{P}(\mathbb{X})$ , that is  $A \cap B \in \mathcal{I}_{\mathbb{X}}$ , then, by (19), we have  $|I_m^{A \cap B}| < \omega$  and, since  $I_m^A \cap I_m^B = I_m^{A \cap B}$ ,  $f(A)$  and  $f(B)$  are incompatible in the poset  $\langle [\omega]^\omega, \subset \rangle$ . Thus (ii) is true as well.

We prove that  $f$  is a surjection. Let  $S \in [\omega]^\omega$  and let  $g : \omega \rightarrow S$  be a bijection. Then  $h = \text{id}_{I \setminus \omega} \cup g : I \rightarrow I$  is an injection. For  $i \in \omega$  we have  $h(i) = g(i) \in S$  and we define  $C_i = X_{g(i)} \in [X_{g(i)}]^{X_i}$ . For  $i \in I \setminus \omega$  let  $C_i = X_i$ . Then, by Claim 4.2,  $C = \bigcup_{i \in I} C_i = \bigcup_{i \in I \setminus \omega} X_i \cup \bigcup_{i \in \omega} X_{g(i)} \in \mathbb{P}(\mathbb{X})$ . Now we have  $f(C) = I_m^C = \{g(i) : i \in \omega\} = S$ .

By Fact 2.5,  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset) \cong \text{sq}([\omega]^\omega, \subset) = (P(\omega)/\text{Fin})^+$ .

Case B:  $|I_m| < \omega$ . Since  $|X| = \omega$  the set  $I = \bigcup_{n \in N} I_n$  is infinite and, hence, there is  $m_0 = \max\{n \in N : |I_n| = \omega\}$ . Clearly we have

$$|I_{m_0}| = \omega \quad \text{and} \quad \forall n \in N \setminus [0, m_0] \quad |I_n| < \omega \quad (20)$$

and  $X = Y \cup Z$ , where  $Y = \bigcup_{n \in N \cap [0, m_0]} \bigcup_{i \in I_n} X_i$  and  $Z = \bigcup_{n \in N \setminus [0, m_0]} \bigcup_{i \in I_n} X_i$ . If  $A \in \mathbb{P}(\mathbb{X})$ , then for each  $n \in N \setminus [0, m_0]$  the copy  $A$  has exactly  $|I_n|$ -many components of size  $n$  and, by (20) and Claim 4.2,  $Z \subset A$ . So, it is easy to see that  $\mathbb{P}(\mathbb{X}) = \{C \cup Z : C \in \mathbb{P}(\mathbb{Y})\}$  and, hence, the mapping  $F : \mathbb{P}(\mathbb{Y}) \rightarrow \mathbb{P}(\mathbb{X})$  given by  $F(C) = C \cup Z$  is well defined and onto. If  $F(C_1) = F(C_2)$  then  $(C_1 \cup Z) \cap Y = (C_2 \cup Z) \cap Y$ , which implies  $C_1 = C_2$ , thus  $F$  is an injection. Clearly  $C_1 \subset C_2$  implies  $F(C_1) \subset F(C_2)$  and, if  $F(C_1) \subset F(C_2)$ , then  $(C_1 \cup Z) \cap Y \subset (C_2 \cup Z) \cap Y$ , which implies  $C_1 \subset C_2$ . Thus  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong_F \langle \mathbb{P}(\mathbb{Y}), \subset \rangle$  and, by Fact 2.2(d),  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset) \cong \text{sq}(\mathbb{P}(\mathbb{Y}), \subset)$ . By (20) the structure  $\mathbb{Y}$  satisfies the assumption of Case A and, hence,  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset) \cong (P(\omega)/\text{Fin})^+$ .

(b) If  $m > 1$ , then there is a partition  $X = A \cup B$  such that  $A \cap X_i \neq \emptyset$  and  $B \cap X_i \neq \emptyset$ , for each  $i \in I_m$ . Now, neither  $A$  nor  $B$  have a component of size  $m$  and, hence, does not contain a copy of  $\mathbb{X}$ . Thus  $\mathbb{X}$  is not indivisible.

If  $m = 1$ , then  $N = \{1\}$  and, since  $\mathbb{P}(\mathbb{X}_i, \mathbb{X}_j) = [\mathbb{X}_j]^{X_i}$ , the structures  $\mathbb{X}_i = \langle \{x_i\}, \rho_{\{x_i\}} \rangle$ ,  $i \in I$ , are isomorphic and, hence, either  $\rho_{\{x_i\}} = \emptyset$ , for all  $i \in I$ , which implies  $\rho = \emptyset$  or  $\rho_{\{x_i\}} = \{\langle x_i, x_i \rangle\}$ , for all  $i \in I$ , which implies  $\rho = \Delta_X$ . Thus, since  $|I| = \omega$ , either  $\mathbb{X} \cong \langle \omega, \emptyset \rangle$  or  $\mathbb{X} \cong \langle \omega, \Delta_\omega \rangle$  and  $\mathbb{P}(\mathbb{X}) = [X]^\omega$  in both cases, which implies that  $\mathbb{X}$  is an indivisible structure.  $\square$

**Case 2:**  $N \not\subset \mathbb{N}$ . Then  $\mu > 0$ ,  $X = (\bigcup_{i \in I \setminus I_\omega} X_i) \dot{\cup} (\bigcup_{i \in I_\omega} X_i) = Y \dot{\cup} Z$  (maybe  $Y = \emptyset$ ) and  $\mathbb{X}$  is the disjoint union of the structures  $\mathbb{Y} = \langle Y, \rho_Y \rangle$  and  $\mathbb{Z} = \langle Z, \rho_Z \rangle$ .

**Claim 4.5 (Case 2.1)** If  $\mu \in \mathbb{N}$ , then

$$(a) \quad \text{sq}(\mathbb{P}(\mathbb{X}), \subset) \cong \begin{cases} ((P(\omega)/\text{Fin})^+)^{\mu} & \text{if } |N_{\text{fin}}| < \omega \text{ and } |Y| < \omega, \\ ((P(\omega)/\text{Fin})^+)^{\mu+1} & \text{if } |N_{\text{fin}}| < \omega \text{ and } |Y| = \omega, \\ \mathbb{P} \times ((P(\omega)/\text{Fin})^+)^{\mu} & \text{if } |N_{\text{fin}}| = \omega, \end{cases} \quad (21)$$

where  $\mathbb{P}$  is an  $\omega_1$ -closed atomless poset;

(b) If  $|N_{\text{fin}}| = \omega$ , then  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  and  $(P(\Delta)/\mathcal{ED}_{\text{fin}})^+ \times (P(\omega)/\text{Fin})^+{}^\mu$  are forcing equivalent posets;

(c)  $\mathbb{X}$  is indivisible iff  $|I| = 1$ , that is  $Y = \emptyset$  and  $\mu = 1$ .

**Proof.** (a) For  $i \in I_\omega$ , let  $A_i, B_i \in [X_i]^\omega$  be disjoint sets,  $A = \bigcup_{i \in I \setminus I_\omega} X_i \cup \bigcup_{i \in I_\omega} A_i$  and  $B = \bigcup_{i \in I \setminus I_\omega} X_i \cup \bigcup_{i \in I_\omega} B_i$ . Then, by Claim 4.2,  $A, B \in \mathbb{P}(\mathbb{X})$  and, since  $A \cap B$  does not contain infinite components, we have  $A \cap B \in \mathcal{I}_{\mathbb{X}}$ . By Facts 3.1 and 2.2(b), the posets  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  and  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  are atomless.

Concerning the closure properties of  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ , first we prove the equality

$$\mathbb{P}(\mathbb{X}) = \{A \cup B : A \in \mathbb{P}(\mathbb{Y}) \wedge B \in \mathbb{P}(\mathbb{Z})\}. \quad (22)$$

If  $C \in \mathbb{P}(\mathbb{X})$ , then, by Claim 4.2, there is an injection  $f : I \rightarrow I$  and there are  $C_i \in [X_{f(i)}]^{|X_i|}$ ,  $i \in I$ , such that  $C = \bigcup_{i \in I} C_i$ . For  $i \in I_\omega$  we have  $C_i \in [X_{f(i)}]^\omega$  and, hence,  $f(i) \in I_\omega$ . Thus  $f[I_\omega] \subset I_\omega$  and, since  $f$  is one-to-one and  $I_\omega$  is finite,  $f[I_\omega] = I_\omega$  and  $f[I \setminus I_\omega] \subset I \setminus I_\omega$ . Now we have  $C = A \dot{\cup} B$ , where  $A = \bigcup_{i \in I \setminus I_\omega} C_i \subset Y$  and  $B = \bigcup_{i \in I_\omega} C_i \subset Z$ . Clearly the structures  $\mathbb{Y}$  and  $\mathbb{Z}$  satisfy the assumptions of Theorem 4.1 and, since the restrictions  $f \upharpoonright I \setminus I_\omega : I \setminus I_\omega \rightarrow I \setminus I_\omega$  and  $f \upharpoonright I_\omega : I_\omega \rightarrow I_\omega$  are injections, by Claim 4.2 we have  $A \in \mathbb{P}(\mathbb{Y})$  and  $B \in \mathbb{P}(\mathbb{Z})$ .

Let  $A \in \mathbb{P}(\mathbb{Y})$  and  $B \in \mathbb{P}(\mathbb{Z})$ . Since the structures  $\mathbb{Y}$  and  $\mathbb{Z}$  satisfy the assumptions of Theorem 4.1, by Claim 4.2 there are injections  $g : I \setminus I_\omega \rightarrow I \setminus I_\omega$  and  $h : I_\omega \rightarrow I_\omega$  and there are  $C_i \in [X_{g(i)}]^{|X_i|}$ ,  $i \in I \setminus I_\omega$ , and  $C_i \in [X_{h(i)}]^{|X_i|}$ ,  $i \in I_\omega$ , such that  $A = \bigcup_{i \in I \setminus I_\omega} C_i$  and  $B = \bigcup_{i \in I_\omega} C_i$ . Now  $f = g \cup h : I \rightarrow I$  is an injection,  $C_i \in [X_{f(i)}]^{|X_i|}$ , for all  $i \in I$ , and, by Claim 4.2,  $A \cup B = \bigcup_{i \in I} C_i \in \mathbb{P}(\mathbb{X})$ . Thus (22) is true.

Now we prove that

$$\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \text{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle \times \text{sq}\langle \mathbb{P}(\mathbb{Z}), \subset \rangle. \quad (23)$$

By (22), the function  $F : \mathbb{P}(\mathbb{Y}) \times \mathbb{P}(\mathbb{Z}) \rightarrow \mathbb{P}(\mathbb{X})$  given by  $F(\langle A, B \rangle) = A \cup B$  is well defined and onto and, clearly, it is a monotone injection. If  $F(\langle A, B \rangle) \subset F(\langle A', B' \rangle)$ , then  $(A \cup B) \cap Y \subset (A' \cup B') \cap Y$ , that is  $A \subset A'$  and, similarly,  $B \subset B'$ , thus  $\langle A, B \rangle \leq \langle A', B' \rangle$ . So  $F$  is an isomorphism and (23) follows from (d) and (f) of Fact 2.2.

If  $|N_{\text{fin}}| < \omega$ , then  $|Y| < \omega$  implies  $|\mathbb{P}(\mathbb{Y})| = 1$  and, hence,  $\text{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle \cong 1$ ; otherwise, if  $|Y| = \omega$ , then, by Claim 4.4,  $\text{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle \cong (P(\omega)/\text{Fin})^+$ . So

$$\text{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle \cong \begin{cases} 1 & \text{if } |N_{\text{fin}}| < \omega \text{ and } |Y| < \omega, \\ (P(\omega)/\text{Fin})^+ & \text{if } |N_{\text{fin}}| < \omega \text{ and } |Y| = \omega. \end{cases} \quad (24)$$

By the assumption, for  $i, j \in I_\omega$  we have  $\mathbb{P}(\mathbb{X}_i, \mathbb{X}_j) = [X_j]^\omega$ . Since  $|I_\omega| < \omega$ , by Claim 4.2 we have  $\mathbb{P}(\mathbb{Z}) = \{\bigcup_{i \in I_\omega} C_i : \forall i \in I_\omega \ C_i \in [X_i]^\omega\}$  which implies

$\langle \mathbb{P}(\mathbb{Z}), \subset \rangle \cong \prod_{i \in I_\omega} \langle [X_i]^\omega, \subset \rangle \cong \langle [\omega]^\omega, \subset \rangle^\mu$ . Since  $\text{sq}\langle [\omega]^\omega, \subset \rangle = (P(\omega)/\text{Fin})^+$ , by (d) and (f) of Fact 2.2 we have

$$\text{sq}\langle \mathbb{P}(\mathbb{Z}), \subset \rangle \cong ((P(\omega)/\text{Fin})^+)^mu. \quad (25)$$

Now, for  $|N_{\text{fin}}| < \omega$  (21) follows from (23), (24) and (25). If  $|N_{\text{fin}}| = \omega$ , then, by Claim 4.3,  $\mathbb{P} = \text{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle$  is  $\omega_1$ -closed atomless and (21) follows from (23) and (25).

(b) By Claim 4.3(e) and Fact 2.2(a), the posets  $\langle \mathbb{P}(\mathbb{Y}), \subset \rangle$ ,  $\text{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle$  and  $(P(\Delta)/\mathcal{ED}_{\text{fin}})^+$  are forcing equivalent. By (23) and (25) we have  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \text{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle \times (P(\omega)/\text{Fin})^+^\mu$ .

(c) Let  $Y = \emptyset$  and  $\mu = 1$ . Then  $\mathbb{P}(\mathbb{X}) = [X]^\omega$  and, clearly,  $\mathbb{X}$  is indivisible.

If  $Y \neq \emptyset$ , then, by (a), each  $C \in \mathbb{P}(\mathbb{X})$  must intersect both  $Y$  and  $Z$  and the partition  $X = Y \cup Z$  witnesses that  $\mathbb{X}$  is not indivisible.

If  $Y = \emptyset$  but  $\mu > 1$ , by (a), each  $C \in \mathbb{P}(\mathbb{X})$  must intersect all components of  $\mathbb{X}$  and for  $i_0 \in I_\omega = I$ , the partition  $X = X_{i_0} \cup \bigcup_{i \in I_\omega \setminus \{i_0\}} X_i$  witnesses that  $\mathbb{X}$  is not indivisible.  $\square$

**Claim 4.6 (Case 2.2)** If  $\mu = \omega$ , then

- (a)  $\mathbb{X}$  is an indivisible structure;
- (b)  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong (P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+$ .

**Proof.** (a) For  $S \subset X$  let  $I_\omega^S = \{i \in I_\omega : |S \cap X_i| = \omega\}$  and first we prove

$$S \in \mathcal{I}_{\mathbb{X}} \Leftrightarrow |I_\omega^S| < \omega. \quad (26)$$

Suppose that  $|I_\omega^S| = \omega$ . Let  $f : I \rightarrow I_\omega^S$  be a bijection. Then, for  $i \in I$  we have  $|S \cap X_{f(i)}| = \omega$  and we can choose  $C_i \in [S \cap X_{f(i)}]^{X_i} \subset \mathbb{P}(\mathbb{X}_i, \mathbb{X}_{f(i)})$ . By Claim 4.2 we have  $C = \bigcup_{i \in I} C_i \in \mathbb{P}(\mathbb{X})$  and, clearly,  $C \subset S$ . Thus  $S \notin \mathcal{I}_{\mathbb{X}}$ .

Let  $S \notin \mathcal{I}_{\mathbb{X}}$  and  $C \in \mathbb{P}(\mathbb{X})$ , where  $C \subset S$ . By Claim 4.2 there are an injection  $f : I \rightarrow I$  and  $C_i \in [X_{f(i)}]^{X_i}$ ,  $i \in I$ , such that  $C = \bigcup_{i \in I} C_i$ . For  $i \in I_\omega$  we have  $C_i \in [X_{f(i)}]^\omega$ , which implies  $|S \cap X_{f(i)}| = \omega$ , that is  $f(i) \in I_\omega^S$ . Thus  $f[I_\omega] \subset I_\omega^S$  and, since  $f$  is one-to-one and  $|I_\omega| = \omega$ , we have  $|I_\omega^S| = \omega$  and (26) is proved.

Suppose that  $\mathbb{X}$  is divisible and  $X = A \cup B$ , where  $A, B \in \mathcal{I}_{\mathbb{X}}$ . Then, by (26),  $|I_\omega^A \cup I_\omega^B| < \omega$  and there is  $i \in I_\omega \setminus (I_\omega^A \cup I_\omega^B)$ . Now,  $|A \cap X_i|, |B \cap X_i| < \omega$ , which is impossible since  $X_i = (A \cap X_i) \cup (B \cap X_i)$  is an infinite set.

(b) W.l.o.g. we suppose that  $I_\omega = \omega$  and  $X_i = \{i\} \times \omega$ , for  $i \in \omega$ . Then  $X = Y \cup (\omega \times \omega)$ , where  $Y = \bigcup_{i \in I \setminus \omega} X_i$ . Clearly, for  $S \subset \omega \times \omega$ ,

$$S \in \text{Fin} \times \text{Fin} \Leftrightarrow |I_\omega^S| < \omega. \quad (27)$$

By (26), for  $A \in \mathbb{P}(\mathbb{X})$  the set  $I_\omega^A = I_\omega^{A \cap (\omega \times \omega)}$  is infinite and by (27) we have  $A \cap (\omega \times \omega) \notin \text{Fin} \times \text{Fin}$ . Hence the mapping

$$f : \langle \mathbb{P}(\mathbb{X}), \subset \rangle \rightarrow \langle (P(\omega \times \omega) / \equiv_{\text{Fin} \times \text{Fin}})^+, \trianglelefteq_{\text{Fin} \times \text{Fin}} \rangle$$

given by  $f(A) = [A \cap (\omega \times \omega)]_{\equiv_{\text{Fin} \times \text{Fin}}}$ , for all  $A \in \mathbb{P}(\mathbb{X})$ , is well defined and we show that it satisfies the assumptions of Fact 2.5. Let  $A, B \in \mathbb{P}(\mathbb{X})$ .

(i) If  $A \subset B$ , then  $(A \cap (\omega \times \omega)) \setminus (B \cap (\omega \times \omega)) = \emptyset \in \text{Fin} \times \text{Fin}$  and  $f(A) = [A \cap (\omega \times \omega)]_{\equiv_{\text{Fin} \times \text{Fin}}} \trianglelefteq_{\text{Fin} \times \text{Fin}} [B \cap (\omega \times \omega)]_{\equiv_{\text{Fin} \times \text{Fin}}} = f(B)$ .

(ii) If  $A$  and  $B$  are incompatible in  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ , then  $A \cap B \in \mathcal{I}_{\mathbb{X}}$  and, by (26),  $|I_\omega^{A \cap B}| < \omega$ , that is  $|I_\omega^{(A \cap (\omega \times \omega)) \cap (B \cap (\omega \times \omega))}| < \omega$ , which, by (27) implies  $(A \cap (\omega \times \omega)) \cap (B \cap (\omega \times \omega)) \in \text{Fin} \times \text{Fin}$ . Hence  $f(A) = [A \cap (\omega \times \omega)]_{\equiv_{\text{Fin} \times \text{Fin}}}$  and  $f(B) = [B \cap (\omega \times \omega)]_{\equiv_{\text{Fin} \times \text{Fin}}}$  are incompatible in  $(P(\omega \times \omega) / \equiv_{\text{Fin} \times \text{Fin}})^+$ .

(iii) We show that  $f$  is a surjection. It is easy to see that for  $A, B \in \mathbb{P}(\mathbb{X})$ ,

$$I_\omega^{A \setminus B} \cup I_\omega^{B \setminus A} = I_\omega^{A \Delta B}. \quad (28)$$

Let  $[S]_{\equiv_{\text{Fin} \times \text{Fin}}} \in (P(\omega \times \omega) / \equiv_{\text{Fin} \times \text{Fin}})^+$ . Then, by (27), we have  $|I_\omega^S| = \omega$ . Let  $g : \omega \rightarrow I_\omega^S$  be a bijection. Then  $h = \text{id}_{I_\omega^S} \cup g : I \rightarrow I$  is an injection. For  $i \in \omega$  we have  $h(i) = g(i) \in I_\omega^S$  and we define  $C_i = S \cap X_{g(i)} \in [X_{g(i)}]^{|X_i|}$ . For  $i \in I \setminus \omega$  let  $C_i = X_i$ . Then, by Claim 4.2,

$$C = \bigcup_{i \in I} C_i = \bigcup_{i \in I \setminus \omega} X_i \cup \bigcup_{i \in \omega} S \cap X_{g(i)} \in \mathbb{P}(\mathbb{X}).$$

Now  $S \setminus C = \bigcup_{j \in \omega \setminus I_\omega^S} S \cap X_j$ , which implies  $I_\omega^{S \setminus C} = \emptyset$  and  $C \setminus S = \bigcup_{i \in I \setminus \omega} X_i \setminus S$ , which implies  $I_\omega^{C \setminus S} = \emptyset$ . So, by (28),  $I_\omega^{C \Delta S} = I_\omega^{(C \cap (\omega \times \omega)) \Delta S} = \emptyset$  and, by (27),  $(C \cap (\omega \times \omega)) \Delta S \in \text{Fin} \times \text{Fin}$ , so  $f(C) = [C \cap (\omega \times \omega)]_{\equiv_{\text{Fin} \times \text{Fin}}} = [S]_{\equiv_{\text{Fin} \times \text{Fin}}}$ .

By Fact 2.5 and since  $\langle (P(\omega \times \omega) / \equiv_{\text{Fin} \times \text{Fin}})^+, \trianglelefteq_{\text{Fin} \times \text{Fin}} \rangle$  is a separative partial order we have  $\text{sq} \langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \text{sq} \langle (P(\omega \times \omega) / \equiv_{\text{Fin} \times \text{Fin}})^+, \trianglelefteq_{\text{Fin} \times \text{Fin}} \rangle \cong \langle (P(\omega \times \omega) / \equiv_{\text{Fin} \times \text{Fin}})^+, \trianglelefteq_{\text{Fin} \times \text{Fin}} \rangle$ .  $\square$

**Proof of Theorem 4.1.** (a) (a4) is Claim 4.6(b). For  $\mu > 0$ , (a1)-(a3) are proved in Claim 4.5(a). For  $\mu = 0$ , (a2) is proved in Claim 4.4(a) and (a3) in Claim 4.3(b). By Facts 2.1 and 2.4,  $\text{sq} \langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is an  $\omega_1$ -closed atomless poset. It is of size  $\mathfrak{c}$  since it contains a reversed binary tree of height  $\omega$  and the set of lower bounds of its branches is of cardinality  $\mathfrak{c}$ . The forcing equivalent of  $\mathbb{P}$  is given in Claim 4.3(e).

(b) follows from (a), Claim 4.5(b) and Fact 2.4.

(c) The implication “ $\Leftarrow$ ” follows from Claims 4.3(a), 4.4(b), 4.5(c) and 4.6(a). For a proof of “ $\Rightarrow$ ” suppose that  $N \notin [\mathbb{N}]^\omega$ ,  $N \neq \{1\}$ ,  $|I| \neq 1$  and  $|I_\omega| < \omega$ .

If  $N \subset \mathbb{N}$ , then, since  $N \notin [\mathbb{N}]^\omega$ , we have  $N = \{n_0, \dots, n_m\}$ , where  $n_0 < \dots < n_m$  and, since  $N \neq \{1\}$ ,  $n_m > 1$ . Let  $x_i \in X_i$ , for  $i \in I_{n_m}$ , let  $A =$

$\bigcup_{i \in I \setminus I_{n_m}} X_i \cup \bigcup_{i \in I_{n_m}} \{x_i\}$  and  $B = \bigcup_{i \in I_{n_m}} X_i \setminus \{x_i\}$ . Then  $X = A \cup B$  and neither  $A$  nor  $B$  contain a copy of  $\mathbb{X}$ , since all their components are of size  $< n_m$ .

If  $N \not\subset \mathbb{N}$ , then  $I_\omega \neq \emptyset$  and, since  $|I_\omega| < \omega$ , we have  $0 < |I_\omega| = m \in \mathbb{N}$ . Since  $|I| \neq 1$ , by Claim 4.5(c)  $\mathbb{X}$  is not indivisible.  $\square$

## 5 Examples

**Example 5.1** Equivalence relations on countable sets. If  $\mathbb{X} = \langle X, \rho \rangle$ , where  $\rho$  is an equivalence relation on a countable set  $X$ , then, clearly, the components  $X_i$ ,  $i \in I$ , of  $\mathbb{X}$  are the equivalence classes determined by  $\rho$  and for each  $i \in I$  the restriction  $\rho_{X_i}$  is the full relation on  $X_i$ , which implies that conditions (i) and (ii) of Theorem 4.1 are satisfied. Thus the poset  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset)$  is  $\omega_1$ -closed and atomless and, hence,  $\mathbb{X}$  belongs to the column  $D$  of Diagram 1. Some examples of such structures are given in Diagram 2, where  $\bigcup_m F_n$  denotes the disjoint union of  $m$  full relations on a set of size  $n$ . We note that  $\mathbb{X}$  is a ultrahomogeneous structure iff

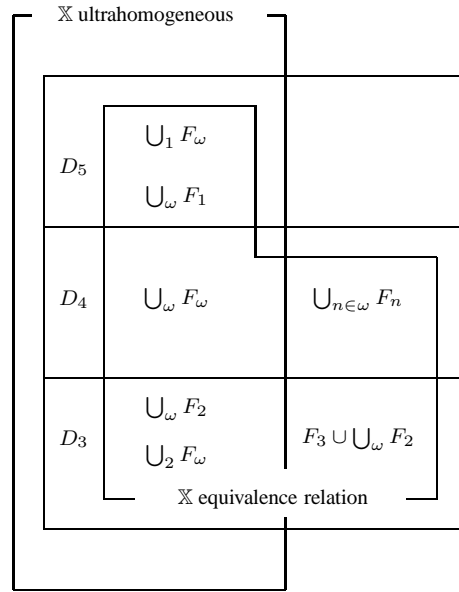


Diagram 2: Equivalence relations on countable sets

all equivalence classes are of the same size, so the following countable equivalence relations are ultrahomogeneous and by Theorem 4.1 have the given properties.

$\bigcup_\omega F_n$ . It is indivisible iff  $n = 1$  (the diagonal) and the poset  $\text{sq}(\mathbb{P}(\mathbb{X}), \subset)$  is isomorphic to  $(P(\omega)/\text{Fin})^+$  which is a  $\mathfrak{t}$ -closed and  $\mathfrak{h}$ -distributive poset.

$\bigcup_n F_\omega$ . It is indivisible iff  $n = 1$  (the full relation) and the poset  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is isomorphic to  $((P(\omega)/\text{Fin})^+)^n$  which is  $\mathfrak{t}$ -closed, but for  $n > 1$  not  $\mathfrak{h}$ -distributive poset in, for example, the Mathias model.

$\bigcup_\omega F_\omega$  (the  $\omega$ -homogeneous-universal equivalence relation). It is indivisible and  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  is isomorphic to  $(P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+$ , which is  $\omega_1$ -closed, but not  $\omega_2$ -closed and, hence, consistently neither  $\mathfrak{t}$ -closed nor  $\mathfrak{h}$ -distributive.

**Example 5.2** Disjoint unions of complete graphs. The same picture as in Example 5.1 is obtained for countable graphs  $\mathbb{X} = \bigcup_{i \in I} \mathbb{X}_i$ , where  $\mathbb{X}_i = \langle X_i, \rho_i \rangle$ ,  $i \in I$ , are disjoint complete graphs (that is  $\rho_i = (X_i \times X_i) \setminus \Delta_{X_i}$ ) since, clearly, conditions (i) and (ii) of Theorem 4.1 are satisfied. Also, by a well known characterization of Lachlan and Woodrow [6] all disconnected countable ultrahomogeneous graphs are of the form  $\bigcup_m K_n$  (the union of  $m$ -many complete graphs of size  $n$ ), where  $mn = \omega$  and  $m > 1$ . So in Diagram 2 we can replace  $F_n$  with  $K_n$ .

**Example 5.3** Disjoint unions of ordinals  $\leq \omega$ . A similar picture is obtained for countable partial orders  $\mathbb{X} = \bigcup_{i \in I} \mathbb{X}_i$ , where  $\mathbb{X}_i$ 's are disjoint copies of ordinals  $\alpha_i \leq \omega$ . (Clearly, linear orders satisfy (ii) of Theorem 4.1 and  $\mathbb{P}(\alpha, \beta) = [\beta]^{|\alpha|}$ , for each two ordinals  $\alpha, \beta \leq \omega$ .) So in Diagram 2 we can replace  $F_n$  with  $L_n$ , where  $L_n \cong n \leq \omega$ , but these partial orderings are not ultrahomogeneous.

**Remark 5.4** All structures analyzed in Examples 5.1, 5.2 and 5.3 are disconnected. But, since  $\mathbb{P}(\langle X, \rho \rangle) = \mathbb{P}(\langle X, \rho^c \rangle)$ , taking their complements we obtain connected structures with the same posets  $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$  and  $\text{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ , having the properties established in these examples. For example, the complement of  $\bigcup_m F_n$  is the graph-theoretic complement of the graph  $\bigcup_m K_n$ .

**Remark 5.5** The structures satisfying the assumptions of Theorem 4.1. Let a countable structure  $\mathbb{X} = \bigcup_{i \in I} \mathbb{X}_i$  satisfy conditions (i) and (ii).

First, (i) implies that all components of the same size are isomorphic.

Second, if  $|X_i| = \omega$  for some  $i \in I$ , then, by (i),  $\mathbb{P}(\mathbb{X}_i) = [X_i]^\omega$  and, by [4],  $\mathbb{X}_i$  is isomorphic to one of the following structures: 1. The empty relation; 2. The complete graph; 3. The natural strict linear order on  $\omega$ ; 4. Its inverse; 5. The diagonal relation; 6. The full relation; 7. The natural reflexive linear order on  $\omega$ ; 8. Its inverse. Thus, since  $\mathbb{X}_i$  is a connected structure, it is isomorphic to the structure 2, 3, 4, 6, 7 or 8 and, by (i) again, this fact implies that

(\*) All  $\mathbb{X}_i$ 's are either full relations or complete graphs or linear orders.

By Claim 4.3(c), (\*) holds when  $\mathbb{X}_i$ 's are finite, but their sizes are unbounded.

But, if the size of the components of  $\mathbb{X}$  is bounded by some  $n \in \mathbb{N}$ , there are structures which do not satisfy (\*). For example, take a disjoint union of  $\omega$  copies of the linear graph  $L_n$  and  $\omega$  copies of the circle graph  $C_{n+1}$ .



## References

- [1] J. Brendle, Cardinal invariants of analytic quotients, a lecture at the Bonn University, 16.6.2009.
- [2] R. Fraïssé, Theory of relations, Revised edition, With an appendix by Norbert Sauer, Studies in Logic and the Foundations of Mathematics, 145, North-Holland, Amsterdam, 2000.
- [3] F. Hernández-Hernández, Distributivity of quotients of countable products of Boolean algebras, *Rend. Istit. Mat. Univ. Trieste* 41 (2009) 27–33 (2010).
- [4] M. S. Kurilić, From  $A_1$  to  $D_5$ : Towards a forcing-related classification of relational structures, submitted.
- [5] M. S. Kurilić, Posets of copies of countable scattered linear orders, submitted.
- [6] A. H. Lachlan, R. E. Woodrow, Countable ultrahomogeneous undirected graphs, *Trans. Amer. Math. Soc.* 262,1 (1980) 51–94.
- [7] M. Pouzet, Application d’une propriété combinatoire des parties d’un ensemble aux groupes et aux relations, *Math. Z.* 150,2 (1976) 117–134.
- [8] S. Shelah, O. Spinas, The distributivity numbers of finite products of  $P(\omega)/\text{fin}$ , *Fund. Math.* 158,1 (1998) 81–93.
- [9] A. Szymański, Zhou Hao Xua, The behaviour of  $\omega^{2^*}$  under some consequences of Martin’s axiom, *General topology and its relations to modern analysis and algebra*, V (Prague, 1981), 577–584, *Sigma Ser. Pure Math.*, 3, Heldermann, Berlin, 1983.